XIX. *On the Electromagnetic Theory of the Reflection and Refraction of Light.*

By GEO. FRAS. FITZGERALD, *M.A., Fellow of Trinity College, Dublin.*

Communicated by G. J. STONEY, *M.A., F.R.S., Secretary of the Queen's University of Ireland.*

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IN the second volume of his 'Electricity and Magnetism' Professor J. CLERK MAXWELL has proposed a very remarkable electromagnetic theory of light, and has worked out the results as far as the transmission of light through uniform crystalline and magnetic media are concerned, leaving the questions of reflection and refraction untouched. These, however, may be very conveniently studied from his point of view.

If we call W the electrostatic energy of the medium, it may be expressed in terms of the electromotive force and the electric displacement at each point as is done in Professor MAXWELL'S 'Electricity and Magnetism,' vol. ii., part iv., ch. 9. I shall adopt his notation and call the electromotive force $\mathfrak{E}$ and its components P, Q, R, and the electric displacement $\mathfrak{D}$ and its components f, g, h. As several of the results of this paper admit of a very elegant expression in Quaternion notation I shall give the work and results in both Cartesian and Quaternion form, confining the German letters to the Quaternion notation. Between these quantities then we have the equation

$$W=-\tfrac{1}{2}\iiint S\mathfrak{E}\mathfrak{D}.dxdydz=\tfrac{1}{2}\iiint(Pf+Qg+Rh)dxdydz$$

Similarly the kinetic energy T may be expressed in terms of the magnetic induction, $\mathfrak{B}$, and the magnetic force, $\mathfrak{H}$, or their components a, b, c and α, β, γ by the equation

$$T=-\frac{1}{8\pi}\iiint .S\mathfrak{B}\mathfrak{H}.dxdydz=\frac{1}{8\pi}\iiint(a\alpha+b\beta+c\gamma)dxdydz$$

I shall at present assume this to be a complete expression for T and return to the case of magnetized media for separate treatment, as Professor MAXWELL has proposed additional terms in this case in order to account for their property of rotatory polarisation. I shall throughout assume the media to be isotropic as regards magnetic induction, for the contrary supposition would enormously complicate the question and be, besides, of doubtful physical applicability. For the present I shall not assume them to be electrostatically isotropic. Hence $\mathfrak{E}$ is a linear vector and self-conjugate function of $\mathfrak{D}$, and consequently P, Q, R linear functions of f, g, h, so that we may write in Quaternion notation

$$\mathfrak{E}=\phi\mathfrak{D}$$

and if we call U the general symmetrical quadratic function of f, g, h we may assume

$$\mathrm{U}=\mathrm{P}f+\mathrm{Q}g+\mathrm{R}h$$

and consequently

$$\mathrm{W}=-\tfrac{1}{2}\iiint \mathrm{S}\mathfrak{D}\phi\mathfrak{D}.dxdydz=\tfrac{1}{2}\iiint \mathrm{U}\,dxdydz$$

As the medium is magnetically isotropic we have

$$\mathfrak{B}=\mu\mathfrak{H} \text{ or } a=\mu\alpha,\ b=\mu\beta,\ c=\mu\gamma$$

where μ is the coefficient of magnetic inductive capacity, and consequently the electrokinetic energy may be written

$$\mathrm{T}=-\frac{\mu}{8\pi}\iiint \mathfrak{H}^2.dxdydz=\frac{\mu}{8\pi}\iiint(\alpha^2+\beta^2+\gamma^2)dxdydz$$

Now I shall assume the mediums to be nonconductors, and although this limits to some extent the applicability of my results, and notably their relation to metallic reflection, yet it is a necessity, for otherwise the problem would be beyond my present powers of solution. With this assumption, and using NEWTON's notation of $\dot{x}$ for $\frac{dx}{dt}$, we have the following equations (see 'Elect. and Mag.,' vol. ii., § 619)

$$4\pi\dot{\mathfrak{D}}=\mathrm{V}\nabla\mathfrak{H}$$

using ∇ for the operation

$$i\frac{d}{dx}+j\frac{d}{dy}+k\frac{d}{dz}$$

or the same in terms of its components, namely,

$$4\pi\dot{f}=\frac{d\gamma}{dy}-\frac{d\beta}{dz}$$

$$4\pi\dot{g}=\frac{d\alpha}{dz}-\frac{d\gamma}{dx}$$

$$4\pi\dot{h}=\frac{d\beta}{dx}-\frac{d\alpha}{dy}$$

Assuming now a quantity $\mathfrak{R}$ with components ξ, η, ζ, such that

$$\mathfrak{R}=\int\mathfrak{H}dt$$

and consequently

$$\dot{\mathfrak{R}}=\mathfrak{H}$$

or in terms of the components

$$\dot{\xi}=\alpha,\ \dot{\eta}=\beta,\ \dot{\zeta}=\gamma$$

we may evidently write

$$4\pi\mathfrak{D}=V\nabla\mathfrak{R}$$

i.e.,

$$4\pi f=\frac{d\zeta}{dy}-\frac{d\eta}{dz},\ 4\pi g=\frac{d\xi}{dz}-\frac{d\zeta}{dx},\ 4\pi h=\frac{d\eta}{dx}-\frac{d\xi}{dy}$$

so that we have

$$\mathrm{W}=-\frac{1}{.32\pi^2}\iiint \mathrm{S}(V\nabla\mathfrak{R}.\phi \mathrm{V}\nabla\mathfrak{R})dxdydz$$

$$\mathrm{T}=-\frac{\mu}{8\pi}\iiint\dot{\mathfrak{R}}^2dxdydz=\frac{\mu}{8\pi}\iiint(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)dxdydz$$

LAGRANGE'S equations of motion may often be very conveniently represented as the conditions that $\int(\mathrm{T}-\mathrm{W})dt$ should be a minimum, or in other words that

$$\delta\int(\mathrm{T}-\mathrm{W})dt=0$$

and this method, from its symmetry, is particularly applicable to the methods of Quaternions.

Proceeding on this method we obtain immediately the equation

$$0=-\int\left[\mu\iiint \mathrm{S}\dot{\mathfrak{R}}\delta\dot{\mathfrak{R}}.dxdydz-\frac{1}{4\pi}\iiint \mathrm{S}(V\nabla\delta\mathfrak{R}.\phi V\nabla\mathfrak{R})dxdydz\right]dt$$

or in Cartesian notation

$$0=\int\left[\frac{\mu}{4\pi}\iiint(\dot{\xi}\delta\dot{\xi}+\dot{\eta}\delta\dot{\eta}+\dot{\zeta}\delta\dot{\zeta})dxdydz-\tfrac{1}{2}\iiint\left(\frac{d\mathrm{U}}{df}.\delta f+\frac{d\mathrm{U}}{dg}.\delta g+\frac{d\mathrm{U}}{dh}.\delta h\right)dxdydz\right]dt$$

Now we may evidently integrate the terms in $\delta\dot{\mathfrak{R}}$ and $\delta\dot{\xi}$, $\delta\dot{\eta}$, $\delta\dot{\zeta}$ with reference to the time, and the terms depending on the limits of the time must vanish separately, and we are not at present concerned with them, so that the equation reduces to

$$\iiiint\left[\mu\mathrm{S}\ddot{\mathfrak{R}}\delta\mathfrak{R}+\frac{1}{4\pi}\mathrm{S}(V\nabla\delta\mathfrak{R}.\phi V\nabla\mathfrak{R})\right]dxdydzdt=0$$

or

$$\iiiint\left[\frac{\mu}{4\pi}(\ddot{\xi}\delta\xi+\ddot{\eta}\delta\eta+\ddot{\zeta}.\delta\zeta)+\tfrac{1}{2}\left(\frac{d\mathrm{U}}{df}.\delta f+\frac{d\mathrm{U}}{dg}.\delta g+\frac{d\mathrm{U}}{dh}\delta h\right)\right]dxdydzdt=0$$

I shall now proceed to integrate this by parts relatively to x, y, z, and in order to express the result conveniently I shall assume ds to be an element of the surface of the medium and $\mathfrak{R}$, with components l, m, n, to be a unit normal to this element of surface, when we evidently obtain

$$\int\left[\frac{1}{4\pi}\iint \mathrm{S}(\mathfrak{R}.\phi V\nabla\mathfrak{R}.\delta\mathfrak{R})ds+\iiint\left\{\mu\mathrm{S}\ddot{\mathfrak{R}}\delta\mathfrak{R}+\frac{1}{4\pi}\mathrm{S}(\mathrm{V}\nabla\phi V\nabla\mathfrak{R}.\delta\mathfrak{R})\right\}dxdydz\right]dt=0$$

or

$$\int\left\{\begin{array}{l}\mu\iiint(\ddot{\xi}\delta\xi+\ddot{\eta}\delta\eta+\ddot{\zeta}\delta\zeta)dxdydz\\ +\frac{1}{2}\iiint\left[\left(\frac{d}{dy}\cdot\frac{d\mathrm{U}}{dh}-\frac{d}{dz}\cdot\frac{d\mathrm{U}}{dg}\right)\delta\xi+\left(\frac{d}{dz}\cdot\frac{d\mathrm{U}}{df}-\frac{d}{dx}\cdot\frac{d\mathrm{U}}{dh}\right)\delta\eta+\left(\frac{d}{dx}\cdot\frac{d\mathrm{U}}{dg}-\frac{d}{dy}\cdot\frac{d\mathrm{U}}{df}\right)\delta\zeta\right]dxdydz\\ +\frac{1}{2}\iint\left[\left(n\frac{d\mathrm{U}}{dg}-m\frac{d\mathrm{U}}{dh}\right)\delta\xi+\left(l\frac{d\mathrm{U}}{dh}-n\frac{d\mathrm{U}}{df}\right)\delta\eta+\left(m\frac{d\mathrm{U}}{df}-l\frac{d\mathrm{U}}{dg}\right)\delta\zeta\right]ds\end{array}\right\}dt=0$$

It is evident that the superficial and general integrals must vanish separately, and as $\delta\mathfrak{R}$ or $\delta\xi$, $\delta\eta$ and $\delta\zeta$, is arbitrary in the general integrals, we must evidently have

$$4\pi\mu\ddot{\mathfrak{R}}+V\nabla\phi V\nabla\mathfrak{R}=0$$

or

$$\mu\ddot{\xi}+\frac{1}{2}\left(\frac{d}{dy}\cdot\frac{d\mathrm{U}}{dh}-\frac{d}{dz}\cdot\frac{d\mathrm{U}}{dg}\right)=0$$

$$\mu\ddot{\eta}+\frac{1}{2}\left(\frac{d}{dz}\cdot\frac{d\mathrm{U}}{df}-\frac{d}{dx}\cdot\frac{d\mathrm{U}}{dh}\right)=0$$

$$\mu\ddot{\zeta}+\frac{1}{2}\left(\frac{d}{dx}\cdot\frac{d\mathrm{U}}{dg}-\frac{d}{dy}\cdot\frac{d\mathrm{U}}{df}\right)=0$$

As it will not take long I will deduce the ordinary equations for the transmission of plane-waves from these before proceeding to discuss the superficial condition. In the first place, as our axes of coordinates are perfectly arbitrary I shall assume z to be normal to the plane of the wave, and consequently ξ, η, ζ to be functions of z and t only, so that $\nabla=k\frac{d}{dz}$ and $\frac{d}{dx}=\frac{d}{dy}=0$, and consequently $\ddot{\zeta}=0$ and $h=0$ and

$$V\nabla\mathfrak{R}=i\frac{d\eta}{dz}-j\frac{d\xi}{dz}$$

so that if

$$\phi\rho=\lambda\mathrm{S}i\rho+\mu\mathrm{S}j\rho+\nu\mathrm{S}k\rho$$

$$\phi V\nabla\mathfrak{R}=\mu.\frac{d\xi}{dz}-\lambda\frac{d\eta}{dz}$$

and consequently

$$4\pi\mu\ddot{\mathfrak{R}}=\mathrm{V}k\mu.\frac{d^2\xi}{dz^2}-\mathrm{V}k\lambda.\frac{d^2\eta}{dz^2}$$

Now if we had originally assumed

$$U=Af^2+Bg^2+Ch^2+2F.gh+2G.hf+2H.fg$$

we should have

$$\lambda=Ai+Hj+Gk,\ \mu=Hi+Bj+Fk$$

so that we obtain

$$4\pi\mu(i\ddot{\xi}+j\ddot{\eta}+k\ddot{\zeta})+i\left(B.\frac{d^2\xi}{dz^2}-H.\frac{d^2\eta}{dz^2}\right)+j\left(A\frac{d^2\eta}{dz^2}-H\frac{d^2\xi}{dz^2}\right)=0$$

so that if we assume i and j parallel to the axes of the section of $S\rho\phi\rho=1$ by the wave-plane we evidently have got $H=0$, and consequently $\ddot{\zeta}=0$ and

$$4\pi\mu\ddot{\xi}=B.\frac{d^2\xi}{dz^2},\ 4\pi\mu\ddot{\eta}=A\frac{d^2\eta}{dz^2}$$

and as A and B are inversely proportional to the squares of the axes of the section of $S\rho\phi\rho=1$ or $U=1$ by the wave-plane, we get M'CULLAGH's result that each component in these directions is propagated independently and with a velocity inversely proportional to the perpendicular axis of the section of $S\rho\phi\rho=1$ by the wave-plane.

This comes out at once from the Cartesian equations, for as $h=0$, U reduces to

$$U=Af+2H.fg+Bg^2$$

and by choosing x and y parallel to the axes of this section, we have $H=0$, so that

$$\frac{dU}{df}=2Af \qquad \frac{dU}{dg}=2B.g$$

$$\text{and } 4\pi f=-\frac{d\eta}{dz} \text{ and } 4\pi g=\frac{d\xi}{dz}$$

and the result of putting these in is evidently the same as before.

Returning now to the superficial conditions, I shall follow M'CULLAGH, and assume that at each point of the surface of separation of two media the values of the elements of the integrals must be equal for the two media, and, indeed, I think it is pretty evident that if the original integrals are to express the whole state of affairs in the case of a motion propagated from one medium into another, the superficial integrals must vanish when the limits introduced in them are the functions corresponding to the two contiguous media.

Using the suffix $_0$ for one medium and $_1$ for the other, and assuming the normal to the surface as k, we get, as δR is evidently arbitrary and the same for both media at the surface of separation,

$$\therefore Vk\phi_0 V\nabla\mathfrak{R}=Vk\phi_1 V\nabla\mathfrak{R}$$

or the two equations

$$\mathrm{S}i\phi_0 V\nabla\mathfrak{R}=\mathrm{S}i\phi_1 V\nabla\mathfrak{R}$$
$$\mathrm{S}j\phi_0 V\nabla\mathfrak{R}=\mathrm{S}j\phi_0 V\nabla\mathfrak{R}$$

which are the same as can be got at once by putting $l=1$ $m=n=0$ in the Cartesian superficial equations, and taking $\delta\xi$ and $\delta\eta$ as arbitrary and independent, when we get

$$\frac{d\mathrm{U}}{df^0}=\frac{d\mathrm{U}}{df^1},\ \frac{d\mathrm{U}}{dg^0}=\frac{d\mathrm{U}}{dg^1}$$

If now we assume that the axis of x is the line of intersection of the plane of incidence and the surface, we may evidently assume $\mathfrak{R}_0$ to be the resultant of an incident and reflected ray, and $\mathfrak{R}_1$ to be the resultant of the two refracted rays, so that we may write

$$\mathfrak{R}_0=j_0\eta_0+j'_0\eta'_0\quad \mathfrak{R}_1=j_1\eta_1+j'_1\eta'_1$$

when j_0, j'_0, j_1, j'_1 are respectively unitvectors parallel to the direction of magnetic displacement in the incident, reflected and each refracted ray. Similarly calling k_0, k'_0, k_1, k'_1 the unitvectors normal to these wave-planes, and z_0, z'_0, z_1, z'_1 the variable distance along these, evidently we may assume $\nabla_0=k_0\frac{d}{dz_0}$, $\nabla'_0=k'_0\frac{d}{dz'_0}$, &c., and substituting these we at once obtain

$$\frac{d\eta_0}{dz_0}.\mathrm{S}i\phi_0 i_0+\frac{d\eta'_0}{dz'_0}.\mathrm{S}i\phi_0 i'_0=\frac{d\eta_1}{dz_1}.\mathrm{S}i\phi_1 i_1+\frac{d\eta'_1}{dz'_1}.\mathrm{S}i\phi_1 i_1$$
$$\frac{d\eta_0}{dz_0}.\mathrm{S}j\phi_0 i_0+\frac{d\eta'_0}{dz'_0}.\mathrm{S}j\phi_0 i'_0=\frac{d\eta_1}{dz_1}.\mathrm{S}j\phi_1 i_1+\frac{d\eta'_1}{dz'_1}.\mathrm{S}j\phi_1 i'_1$$

In these I shall now assume $\phi_0=1$ as it is convenient to suppose this medium to be isotropic and the velocity of propagation in it unity, and i_0 and i'_0 are the directions in the wave-plane perpendicular to the magnetic displacement; so that if α_0, β_0, γ_0, α'_0, β'_0, γ'_0 be the direction angles of these lines referred to the superficial axes

$$\mathrm{S}i\phi_0 i_0=\cos\alpha_0,\ \mathrm{S}i\phi_0 i'_0=\cos\alpha'_0,\ \mathrm{S}j\phi_0 i_0=\cos\beta_0,\ \mathrm{S}j\phi_0 i'_0=\cos\beta'_0,$$

and if s^{-1} be that axis of the section of $\mathrm{S}\rho\phi\rho=1$ by one of the refracted wave-planes which is perpendicular to the direction of magnetic displacement, it is evidently the velocity of propagation of this wave in this medium, and we may write

$$i_1=\mathrm{U}s=s^{-1}.\mathrm{T}s\ .\,\because\ \phi i_1=\phi s^{-1}.\mathrm{T}s=v\mathrm{T}s$$

when v^{-1} is the perpendicular on the corresponding tangent plane, and consequently

$$\mathrm{S}i\phi i_1=\mathrm{T}v\mathrm{T}s.\ \cos\alpha_1$$

when α_1 is the angle between v and the axis of x, and evidently Ts represents the velocity of this wave, while Vvj is the direction of the ray. Hence our equations may be written

$$\cos \alpha_0 \frac{d\eta_0}{dz_0} + \cos \alpha'_0 \frac{d\eta'_0}{dz'_0} = \mathrm{T}v.\mathrm{T}s.\, \cos \alpha_1 \frac{d\eta_1}{dz_1} + \mathrm{T}v'\mathrm{T}s'.\, \cos \alpha'_1 . \frac{d\eta'_1}{dz'_1}$$

$$\cos \beta_0 . \frac{d\eta_0}{dz_0} + \cos \beta'_0 \frac{d\eta'_0}{dz'_0} = \mathrm{T}v.\mathrm{T}s.\, \cos \beta_1 \frac{d\eta_1}{dz_1} + \mathrm{T}v'\mathrm{T}s'.\, \cos \beta'_1 \frac{d\eta'_1}{dz'_1}$$

These may be readily deduced from the Cartesian equations as follows: U_0 being isotropic it can be written $= \mathrm{A}_0(f_0^2+g_0^2+h_0^2)$, and is evidently unaltered by transformation, and if $l : m : n$ and $l_1 : m_1 : n_1$ be the direction cosines of x and y to any arbitrary axis, we get

$$\mathrm{A}_0 f_0 = l \frac{d\mathrm{U}_1}{df} + m \frac{d\mathrm{U}_1}{dg} + n \frac{d\mathrm{U}_1}{dh}$$

$$\mathrm{A}_0 g_0 = l_1 \frac{d\mathrm{U}_1}{df} + m_1 \frac{d\mathrm{U}_1}{dg} + n_1 \frac{d\mathrm{U}_1}{dh}$$

and now, as these are linear, we may suppose the superficial disturbance in one medium to be due to an incident and reflected wave, and in the other to two refracted rays, so that we can write these as

$$\mathrm{A}_0(f_0+f'_0) = \left(l \frac{d\mathrm{U}_1}{df} + m \frac{d\mathrm{U}_1}{dg} + n \frac{d\mathrm{U}_1}{dh}\right) + \left(l' \frac{d\mathrm{U}'_1}{df} + m' \frac{d\mathrm{U}'_1}{dg} + n' \frac{d\mathrm{U}'_1}{dh}\right)$$

$$\mathrm{A}_0(g_0+g'_0) = \left(l_1 \frac{d\mathrm{U}_1}{df} + m_1 \frac{d\mathrm{U}_1}{dg} + n_1 \frac{d\mathrm{U}}{dh_1}\right) + \left(l'_1 \frac{d\mathrm{U}'_1}{df_1} + m'_1 \frac{d\mathrm{U}'_1}{dg_1} + n'_1 \frac{d\mathrm{U}'_1}{dh}\right)$$

and as U_1 and U'_1 are supposed to be referred to arbitrary axes, we may suppose U_1 to be referred to such that η_1 is the only component—*i.e.*, to such that the direction of the magnetic force is the axis of y, that of z being normal to the wave-plane, and similarly η'_1 being the only component in U'_1 while the z axis in this case is normal to its wave-plane, and of course this η_1 will be a function of z_1 only and η'_1 of z'_1 only, these being the corresponding ordinates. We thus obtain

$$4\pi f_1 = -\frac{d\eta_1}{dz_1} \quad 4\pi f'_1 = -\frac{d\eta'_1}{dz'_1}$$

$$g_1 = g'_1 = h_1 = h'_1 = 0$$

so that

$$\frac{d\mathrm{U}}{df_1} = \mathrm{A}_1 f_1 \quad \frac{d\mathrm{U}'}{df'_1} = \mathrm{A}'_1 f'_1 \quad \frac{d\mathrm{U}_1}{dg} = \mathrm{H}_1 f_1 \quad \frac{d\mathrm{U}'_1}{dg} = \mathrm{H}'_1 f'_1 \quad \frac{d\mathrm{U}_1}{dh} = \mathrm{G}_1 f_1 \quad \frac{d\mathrm{U}'_1}{df'_1} = \mathrm{G}'_1 f'_1$$

and the equations become

$$\mathrm{A}_0(f_0+f'_0)=(l\mathrm{A}_1+m\mathrm{H}_1+n\mathrm{G}_1)\frac{d\eta'}{dz_1}+(l'\mathrm{A}'_1+m'\mathrm{H}'_1+n'\mathrm{G}'_1)\,\frac{d\eta'_1}{dz'_1}$$

$$\mathrm{A}_0(g_0+g'_0)=(l_1\mathrm{A}_1+m_1\mathrm{H}_1+n_1\mathrm{G}_1)\frac{d\eta_1}{dz_1}+(l'_1\mathrm{A}'_1+m'_1\mathrm{H}'_1+n'_1\mathrm{G}'_1)\,\frac{d\eta'_1}{dz'_1}$$

Now if we consider A_1, H_1, G_1 we see that they are proportional to the direction cosines of the perpendicular on the tangent plane to U where it is met by $g=0$, $h=0$ —*i.e.*, it is the direction conjugate to the electric displacement corresponding to η_1. Now if we call $\mathrm{A}_0 s$ the velocity of propagation of this wave and $\mathrm{A}_0 s'$ of the other, we see at once that s^{-1} is an axis of the section of U=a constant by one wave-plane, and s'^{-1} is an axis of the section by the other wave-plane, and if v^{-1} and v'^{-1} be the corresponding conjugate directions, and α_1 and α'_1, β_1 and β'_1 the angles between these latter and the superficial x and y axes, and if α_0, β_0, α'_0, β'_0 be the corresponding angles between η_0 and η'_0 and these same superficial axes, we get exactly the same equations as before, and write

$$\frac{d\eta_0}{dz_0}\,.\cos\alpha_0+\frac{d\eta'_0}{dz'_0}\,.\cos\alpha'_0=vs\,.\cos\alpha_1\frac{d\eta_1}{dz_1}+v's'\,.\cos\alpha'_1\frac{d\eta'_1}{dz'_1}$$

$$\frac{d\eta_0}{dz_0}\,.\cos\beta_0+\frac{d\eta'_0}{dz'_0}\,.\cos\beta'_0=vs\,.\cos\beta_1\frac{d\eta_1}{dz_1}+v's'\,.\cos\alpha'_1\frac{d\eta'_1}{dz'_1}$$

In order to reduce these further we assume

$$\eta_0=\mathrm{T}_0\,.\cos\frac{2\pi}{\lambda_0}\,.\,(t-z_0)\qquad\eta'_0=\mathrm{T}'_0\cos\frac{2\pi}{\lambda'_0}(t-z'_0)$$

$$\eta_1=\mathrm{T}_1\cos\frac{2\pi}{\lambda_1}\,(st-z_1)\qquad\eta'_1=\mathrm{T}'_1\cos\frac{2\pi}{\lambda'_1}(s't-z'_1)$$

and if our superficial axes are so placed that the axis of x is the intersection of the plane of incidence and the surface, and if i_0, i'_0, i_1, i'_1 be the angles the respective wave normals make with the normal to the surface, we evidently may write

$$z_0=z\cos i+x\sin i\qquad z'_0=z\cos i'+z\sin i'$$

$$z_1=z\cos i_1+x\sin i_1\qquad z'_1=z\cos i'_1+z\sin i'_1$$

as it is easy to convince oneself that any terms involving y would be inadmissible, as they could not by hypothesis occur in z_0, and the terms involving the time could not vanish out of our equations if they occurred in the others. Hence these wave normals are all in the same plane. When $z=0$ our equations must evidently be true independently of the time and x, from which we see that no change of phase is possible in reflection. Hence these equations cannot explain metallic reflection. Indeed, this question of change of phase seems to be one of a higher order than I am here dealing with, and requires a discussion of the nature of the transition from one medium to another, which, of course, cannot be abrupt, as our equations suppose, nor indeed probably, in these cases, even very small compared with the vibrations. In order that these terms involving the time should disappear from the equations when $z=0$ we must have

$$\lambda_0 : \lambda'_0 : \lambda_1 : \lambda'_1 :: \sin i_0 : \sin i'_0 : \sin i_1 : \sin i'_1 :: 1 : 1 : s : s'$$

which involves that the angle of incidence should be equal to the angle of reflection; and if the second medium were an ordinary one, so that $s=\frac{1}{\rho}=s'$ we should have that the ratio of the sines of the angles of incidence and refraction was constant. Putting in these values, our equations reduce to

$$T_0 \cos\alpha_0 + T'_0 \cos\alpha'_0 = vT_1 \cos\alpha_1 + v'T'_1 \cos\alpha'_1$$
$$T_0 \cos\beta_0 + T'_0 \cos\beta'_0 = vT_1 \cos\beta_1 + v'T'_1 \cos\beta'_1$$

together with the condition that when $z=0$ the superficial displacements should be the same to whichever medium they belong, namely,

$$\xi=\xi',\ \eta=\eta',\ \zeta=\zeta'$$

As each of these is a resolved part of the vibrations η_0, η'_0 and η_1, η'_1 we get three additional equations, the last of which, however, is the same as the second of the former ones and there result, consequently, but four equations from which the four quantities, namely, the three intensities T'_0, T_1, T'_1, and the azimuth of T'_0 are to be determined. It is remarkable that whether we assumed or no that $\zeta=\zeta'$ it is here introduced. That is, however, no proof that it is wrong to omit it, as in FRESNEL'S method of obtaining the intensities of the reflected and refracted rays, for the fact of its turning up independently shows that there is something at least debateable about it, and as I shall have cause to omit this equation as leading to inconvenient results in a subsequent part of my paper, I thought it well to mention that there is something curious about it even here. These equations are those long ago given by M'CULLAGH in the 'Transactions of the Royal Irish Academy,' vol. xxi., to solve the problem of crystalline reflection and refraction, and from which he deduces his beautiful theorem of the polar plane and thus marvellously simplifies an extremely complicated problem. Indeed, as the forms into which I have thrown T and W are identical with his expressions for what are practically the same quantities, my whole investigation so far is but a modification of his. In the simple case of the second medium being also isotropic we have that $v=v'=s=s'$, and if we in the first place suppose η_0 to be in the plane of incidence, we have at once $\alpha_0=90°$, and consequently $\alpha'_0=\alpha'=90°$, and T'_1

vanishes, or at least may be supposed to do so, the medium being isotropic and $s=s'$, so that the two refracted waves coincide. Also $\beta_0=0$ and $\beta'_0=\beta_1=0$ likewise, as can easily be seen by assuming that the reflected and refracted waves have components out of the plane of incidence and then trying to satisfy the equations. Hence our equations reduce to

$$\mathrm{T}_0+\mathrm{T}'_0=\frac{1}{\rho^*}\mathrm{T}_1$$

and $\xi_0=\xi'_0$ becomes

$$(\mathrm{T}_0-\mathrm{T}'_0)\sin i=\mathrm{T}_1\cos r$$

i being the angle of incidence and r of reflection. The first of these is

$$(\mathrm{T}_0+\mathrm{T}'_0)\sin i=\mathrm{T}_1\sin r$$

and solving them we get

$$\mathrm{T}'_0=-\mathrm{T}_0\frac{\sin(i-r)}{\sin(i+r)}\quad \mathrm{T}_1=\mathrm{T}_0\cdot\frac{\sin 2i}{\sin(i+r)}$$

If η_0 be perpendicular to the plane of incidence we obtain $\alpha_0=i$, $\beta_0=0$, and our equations become

$$\mathrm{T}_0+\mathrm{T}'_0=\mathrm{T}_1$$

and

$$(\mathrm{T}_0-\mathrm{T}'_0)\sin i\cos i=\mathrm{T}_1\sin r\cos r$$

which give

$$\mathrm{T}_0=-\mathrm{T}'_0\cdot\frac{\tan(i-r)}{\tan(i+r)}\quad \mathrm{T}'_1=\mathrm{T}_0\frac{\sin 2i}{\sin(i+r)\cos(i-r)}$$

Having now deduced the already known laws of reflection and refraction of light at crystalline and ordinary surfaces, I shall proceed to consider the case that Mr KERR's wonderfully beautiful experiments have made so interesting—namely, the case of reflection from magnetic surfaces. In order to do this I shall assume, with Professor J. CLERK MAXWELL (see 'Electricity and Magnetism,' vol. ii., § 824), that the kinetic energy of the medium contains a term depending on the displacement of certain supposed vortices, and that it may be expressed by an equation of the form (§ 826)

$$\mathrm{T}=-\mathrm{C}\iiint\mathrm{S}\left(\frac{d\mathfrak{R}}{d\theta}\cdot V\nabla\dot{\mathfrak{R}}\right)dxdydz=4\pi\mathrm{C}\iiint\left(\frac{d\xi}{d\theta}\cdot\dot{f}+\frac{d\eta}{d\theta}\cdot\dot{g}+\frac{d\zeta}{d\theta}\dot{h}\right)dxdydz$$

where

$$\frac{d}{d\theta}=\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}$$

and α, β, γ are now the components of the vortex $\mathfrak{M}$.

I shall assume the medium to be isotropic, so that taking $\phi=\frac{4\pi}{\mathrm{K}}$ the electrostatic energy of the medium may be expressed as

---

* When ρ is the refractive index.

$$\mathrm{W}=-\frac{1}{8\pi\mathrm{K}}\iiint(V\nabla\mathfrak{R})^2.dxdydz$$

$$=+\frac{1}{8\pi\mathrm{K}}\iiint\left[\left(\frac{d\xi}{dy}-\frac{d\eta}{dz}\right)^2+\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)^2+\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)^2\right]dxdydz$$

While the complete value of T is

$$\mathrm{T}=-\frac{1}{8\pi}\iiint\left[\mu\dot{\mathfrak{R}}^2+8\pi\mathrm{CS}\left(\frac{d\mathfrak{R}}{d\theta}.V\nabla\dot{\mathfrak{R}}\right)\right]dxdydz$$

$$=\cdot\frac{1}{8\pi}\iiint\left[\mu(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)+8\pi\mathrm{C}\left\{\frac{d\xi}{d\theta}\left(\frac{d\dot{\zeta}}{dy}-\frac{d\dot{\eta}}{dz}\right)+\frac{d\eta}{d\theta}\left(\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)+\frac{d\zeta}{d\theta}\left(\frac{d\dot{\eta}}{dx}-\frac{d\dot{\xi}}{dy}\right)\right\}\right]dxdydz$$

Working as before, and obtaining our equations as the condition that

$$\int(\mathrm{T}-\mathrm{W})dt$$

should be a minimum, we have to satisfy the equations

$$\iiiint\left[\mu\mathrm{S}\dot{\mathfrak{R}}\delta\dot{\mathfrak{R}}+4\pi\mathrm{CS}\left(\frac{d\delta\mathfrak{R}}{d\theta}.V\nabla\dot{\mathfrak{R}}\right)+4\pi\mathrm{CS}\left(\frac{d\mathfrak{R}}{d\theta}.V\nabla\delta\dot{\mathfrak{R}}\right)-\frac{1}{\mathrm{K}}\mathrm{S}(V\nabla\mathfrak{R}.V\nabla\delta\mathfrak{R})\right]dxdydzdt=0$$

or in Cartesian coordinates

$$\iiiint\left\{\begin{aligned}&\mu(\dot{\xi}\delta\dot{\xi}+\dot{\eta}\delta\dot{\eta}+\dot{\zeta}\delta\dot{\zeta})\\&+4\pi\mathrm{C}\left\{\frac{d\delta\xi}{d\theta}\left(\frac{d\dot{\zeta}}{dy}-\frac{d\dot{\eta}}{dz}\right)+\frac{d\delta\eta}{d\theta}\left(\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)+\frac{d\delta\zeta}{d\theta}\left(\frac{d\dot{\eta}}{dx}-\frac{d\dot{\xi}}{dy}\right)\right\}\\&+4\pi\mathrm{C}\left\{\frac{d\xi}{d\theta}\left(\frac{d\delta\dot{\zeta}}{dy}-\frac{d\delta\dot{\eta}}{dz}\right)+\frac{d\delta\eta}{d\theta}\left(\frac{d\delta\dot{\xi}}{dz}-\frac{d\delta\dot{\zeta}}{dx}\right)+\frac{d\zeta}{d\theta}\left(\frac{d\delta\dot{\eta}}{dx}-\frac{d\delta\dot{\xi}}{dy}\right)\right\}\\&-\frac{1}{\mathrm{K}}\left\{\left(\frac{d\delta\zeta}{dy}-\frac{d\delta\eta}{dy}\right)\left(\frac{d\zeta}{dy}-\frac{d\eta}{dy}\right)+\left(\frac{d\delta\xi}{dz}-\frac{d\delta\zeta}{dx}\right)\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)\right.\\&\qquad\qquad\left.+\left(\frac{d\delta\eta}{dx}-\frac{d\delta\xi}{dy}\right)\left(\frac{d\eta}{dx}-\frac{d\zeta}{dy}\right)\right\}\end{aligned}\right\}dxdydzdt=0$$

Now of course the terms depending on C are the only additional ones, and it will be necessary to integrate them both relatively to the time and also relatively to θ. The integration with respect to the time is easily performed as it merely consists in removing the dot from one of the terms under the integral to the other and changing the sign, so that, neglecting the terms depending on the limits of the time with which we are not concerned, our equation becomes

$$\iiiint\left[\mu\mathrm{S}(\ddot{\mathfrak{R}}\delta\mathfrak{R})-4\pi\mathrm{CS}\frac{d\delta\mathfrak{R}}{d\theta}V\nabla\dot{\mathfrak{R}}+4\pi\mathrm{CS}\left(\frac{d\dot{\mathfrak{R}}}{d\theta}V\nabla\delta\mathfrak{R}\right)+\frac{1}{\mathrm{K}}\mathrm{S}(V\nabla\mathfrak{R}.V\nabla\delta\mathfrak{R})\right]dxdydzdt=0$$

or in Cartesian coordinates

$$\iiiint\left\{\begin{array}{l} \mu(\ddot{\xi}\delta\xi+\ddot{\eta}\delta\eta+\ddot{\zeta}\delta\zeta) \\ -4\pi\mathrm{C}\left\{\frac{d\delta\xi}{d\theta}\cdot\left(\frac{d\dot{\zeta}}{dy}-\frac{d\dot{\eta}}{dz}\right)+\frac{d\delta\eta}{d\theta}\cdot\left(\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)+\frac{d\delta\zeta}{d\theta}\cdot\left(\frac{d\dot{\eta}}{dx}-\frac{d\xi}{dy}\right)\right\} \\ +4\pi\mathrm{C}\left\{\frac{d\dot{\xi}}{d\theta}\left(\frac{d\delta\zeta}{dy}-\frac{d\delta\eta}{dz}\right)+\frac{d\dot{\eta}}{d\theta}\left(\frac{d\delta\xi}{dz}-\frac{d\delta\zeta}{dx}\right)+\frac{d\dot{\zeta}}{d\theta}\left(\frac{d\delta\dot{\eta}}{dx}-\frac{d\delta\xi}{dy}\right)\right\} \\ +\frac{1}{\mathrm{K}}\left\{\left(\frac{d\delta\zeta}{dy}-\frac{d\delta\eta}{dz}\right)\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)+\left(\frac{d\delta\xi}{dz}-\frac{d\delta\zeta}{dx}\right)\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)\right. \\ \qquad\qquad\qquad\qquad \left.+\left(\frac{d\delta\eta}{dx}-\frac{d\delta\xi}{d\eta}\right)\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right\} \end{array}\right\}dxdydzdt=0$$

Integrating these now by parts relatively to x, y, z, and calling $l:m:n$ the direction cosines of $\mathfrak{N}$ the normal to the surface of separation, and $\alpha l+\beta m+\gamma n=\mathrm{S}\mathfrak{M}\mathfrak{N}=\epsilon$ when $\mathfrak{M}$ is the vortex whose components are α, β, γ, we get

$$\int\left[\iint\left\{\mathrm{S}.\left[4\pi\mathrm{C}\left\{\mathrm{V}\left(\mathfrak{N}.\frac{d\dot{\mathfrak{R}}}{d\theta}\right)-\epsilon.\mathrm{V}\nabla\mathfrak{R}\right\}+\frac{1}{\mathrm{K}}\mathrm{V}(\mathfrak{N}.\mathrm{V}\nabla\mathfrak{R})\right]\delta\mathfrak{R}\right\}ds\right.$$
$$\left.+\iiint\mathrm{S}.\left\{\left(\mu\ddot{\mathfrak{R}}+8\pi\mathrm{CV}\nabla\frac{d\dot{\mathfrak{R}}}{d\theta}+\frac{1}{\mathrm{K}}\mathrm{V}\nabla\mathrm{V}\nabla\mathfrak{R}\right)\delta\mathfrak{R}\right\}dxdydz\right]dt=0$$

or in Cartesian coordinates

$$\int\left\{\begin{array}{l} \iint\frac{1}{\mathrm{K}}\left[\left\{n\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)-m\left(\frac{d\eta}{dz}-\frac{d\xi}{dy}\right)\right\}\delta\xi+\left\{l\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)-n\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)\right\}\delta\eta\right. \\ \qquad\qquad\qquad\qquad \left.+\left\{m\left(\frac{d\xi}{dy}-\frac{d\eta}{dz}\right)-l\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)\right\}\delta\zeta\right]ds \\ +4\pi\mathrm{C}\iint\left\{\frac{d}{d\theta}(n\dot{\eta}-m\dot{\zeta}).\delta\xi+\frac{d}{d\theta}(l\dot{\zeta}-n\dot{\xi}).\delta\eta+\frac{d}{d\theta}(m\dot{\xi}-l\dot{\eta}).\delta\zeta\right\}ds \\ -4\pi\mathrm{C}\iint(l\alpha+m\beta+n\gamma)\left\{\left(\frac{d\dot{\zeta}}{dy}-\frac{d\dot{\eta}}{dz}\right)\delta\xi+\left(\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)\delta\eta+\left(\frac{d\dot{\eta}}{dx}-\frac{d\dot{\xi}}{dy}\right)\delta\zeta\right\}ds \\ +\iiint\mu(\ddot{\xi}\delta\xi+\ddot{\eta}\delta\eta+\ddot{\zeta}\delta\zeta)dxdydz \\ +8\pi\mathrm{C}\iiint\left\{\frac{d}{d\theta}\left(\frac{d\dot{\zeta}}{dy}-\frac{d\dot{\eta}}{dz}\right).\delta\xi+\frac{d}{d\theta}\left(\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)\delta\eta+\frac{d}{d\theta}\left(\frac{d\dot{\eta}}{dx}-\frac{d\dot{\xi}}{dy}\right)\delta\zeta\right\}dxdydz \\ +\frac{1}{\mathrm{K}}\iiint\left\{\left[\frac{d}{dy}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)-\frac{d}{dz}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)\right]\delta\xi+\left[\frac{d}{dz}\left(\frac{d\xi}{dy}-\frac{d\eta}{dz}\right)-\frac{d}{dx}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right]\delta\eta\right. \\ \qquad\qquad\qquad\qquad \left.+\left[\frac{d}{dx}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)-\frac{d}{dy}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)\right]\delta\zeta\right\}dxdydz \end{array}\right\}dt=0$$

I shall not consider the general equations of wave propagation in the Cartesian form as they are the same as those given by MAXWELL in a more general form, for he assumes that the static energy of the medium is expressed more generally than I have assumed, but as the Quaternion investigation is not long I shall give it.

The equation of motion is

$$\mu\ddot{\mathfrak{R}}+8\pi \mathrm{C} \mathrm{V} \nabla \frac{d\dot{\mathfrak{R}}}{d\theta}+\frac{1}{\mathrm{K}} \mathrm{V} \nabla \mathrm{V} \nabla \mathfrak{R}=0$$

and if z be the normal to a wave-plane, we may evidently assume $\mathrm{V}=k\frac{d}{dz}$, and as $\mathfrak{R}$ is in the wave-plane we have

$$\mathfrak{R}=ai \sin \frac{2\pi}{\lambda}(vt-z)+bj \cos \frac{2\pi}{\lambda}(vt-z)$$

and as $\frac{d}{d\theta}=\gamma\frac{d}{dz}$ our equation becomes

$$\mu\ddot{\mathfrak{R}}+8\pi \mathrm{C}\gamma . \mathrm{V}k\frac{d^3\mathfrak{R}}{dz^2dt}+\frac{1}{\mathrm{K}} . \mathrm{V}k\frac{d}{dz}\mathrm{V}k\frac{d\mathfrak{R}}{dz}=0$$

Substituting for $\mathfrak{R}$, and equating the coefficients of i and j separately to zero, we get

$$a\mu v^2-\frac{16\pi^2\mathrm{C}\gamma}{\lambda^2} . bv-\frac{a}{\mathrm{K}}=0, \quad b\mu v^2-\frac{16\pi^2\mathrm{C}\gamma}{\lambda} . av-\frac{b}{\mathrm{K}}=0$$

from which it is easy to see that $a=\pm b$ and that v is determined by the equation

$$\mu v^2 \mp \frac{16\pi^2\mathrm{C}\gamma}{\lambda^2} . v-\frac{1}{\mathrm{K}}=0$$

which gives of course two different values of v, one for each circularly polarised ray, or disregarding the solutions for waves going in the negative direction, we have approximately

$$v_1=\frac{1}{\sqrt{\mu\mathrm{K}}}+\frac{8\pi^2\mathrm{C}\gamma}{\mu\lambda^2}, \quad v_2=\frac{1}{\sqrt{\mu\mathrm{K}}}-\frac{8\pi^2\mathrm{C}\gamma}{\lambda^2\mu}$$

and of course C can easily be determined from these by observing the rotation produced, but there is nothing except experiment to prove that C may not be a function of λ, as we know K to be, to some extent at least; so that using these formulæ in order to obtain the laws of dispersion of rotatory polarisation seems to approach towards deducing the known from the unknown. All we can be sure of is that C is in general extremely minute.

Returning to the superficial equations, and using / as a sign of substitution, we get

$$\Big/\frac{1}{\mathrm{K}}.\mathrm{V}(\mathfrak{R}.V\nabla\,\mathfrak{R})+4\pi\mathrm{C}\left(\mathrm{V}\mathfrak{R}\frac{d\dot{\mathfrak{R}}}{d\theta}-\epsilon\mathrm{V}\nabla\,\dot{\mathfrak{R}}\right)=0$$

As I intend only to work out the results in Cartesian coordinates, I shall confine myself to them, and for simplification assume the axes to be z normal to and x and y in the surface, and consequently $l=m=0$, $n=1$, and $\epsilon=\gamma$. C may also be supposed to vanish for one of the media for which K_1 is the dielectric inductive capacity. I shall also assume that $\delta\zeta=0$ as the vanishing of its coefficient leads to inconvenient results, and the assumption may be to some extent justified by considering that as these $\delta\xi$, $\delta\eta$, $\delta\zeta$ are superficial values, no virtual displacement out of the surface, as this would be, is admissible. From the other two, $\delta\xi$ and $\delta\eta$, we evidently get

$$\frac{1}{\mathrm{K}_1}\left(\frac{d\xi_1}{dz}-\frac{d\zeta_1}{dx}\right)=\frac{1}{\mathrm{K}}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+4\pi\mathrm{C}\frac{d\dot{\eta}}{d\theta}-4\pi\mathrm{C}\gamma\left(\frac{d\dot{\zeta}}{dy}-\frac{d\dot{\eta}}{dz}\right)$$

$$\frac{1}{\mathrm{K}_1}\left(\frac{d\zeta_1}{dy}-\frac{d\eta_1}{dz}\right)=\frac{1}{\mathrm{K}}\cdot\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)+4\pi\mathrm{C}\frac{d\dot{\xi}}{d\theta}+4\pi\mathrm{C}\gamma.\left(\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)$$

A remarkable point about these equations is that they admit of being integrated with regard to the time as far as their right-hand members are concerned, so that in addition to the values which satisfy them as they stand, and which are the only ones of much interest, there are other periodic values of ξ, η, ζ independently of ξ_1, η_1, ζ_1 which satisfy these superficial conditions, and which may consequently be looked upon as a sort of free vibration of the surface of the medium. But even if this could be propagated into the rest of the ether it is improbable that the resultant vibrations would be of such a period as to be visible, though some energy might be expended on them.

I shall now further assume that the axis of x is the intersection of the surface with the plane of incidence of a plane-wave, and that consequently none of my quantities are functions of y, which reduces these equations to

$$\frac{1}{\mathrm{K}_1}\left(\frac{d\xi_1}{dz}-\frac{d\zeta_1}{dx}\right)=\frac{1}{\mathrm{K}}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+4\pi\mathrm{C}\left(\frac{d\dot{\eta}}{d\theta}+\gamma_1\frac{d\dot{\eta}}{dz}\right)$$

$$\cdot\frac{1}{\mathrm{K}_1}\cdot\frac{d\eta_1}{dz}=\frac{1}{\mathrm{K}}\cdot\frac{d\eta}{dz}-4\pi\mathrm{C}\left\{\frac{d\dot{\xi}}{d\theta}+\gamma\left(\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)\right\}$$

In order still further to simplify the problem I shall now consider separately the cases in which the magnetisation of the medium is normal and that in which it is in the surface. In the first place, it is evident that if it were all in y we should have $\alpha=\gamma=0$, $\beta=\mathfrak{M}$, and the coefficient of C would vanish in both equations because

$\frac{d}{d\theta}=\mathfrak{M}\frac{d}{d\eta}$, and hence we get Mr. KERR's result ('Phil. Mag.,' March, 1878, p. 174) that when the plane of incidence is normal to the lines of magnetic force the magnetising of the mirror produces no change in the reflected light.

Assuming, then, first that the magnetisation is normal to the surface, we have $\alpha=\beta=0$, $\gamma=\mathfrak{M}$, and $\frac{d}{d\theta}=\mathfrak{M}\frac{d}{dz}$, so that calling $4\pi\mathrm{CK}_1\mathfrak{M}=\nu$ our equations become

$$\left.\begin{aligned}\frac{d\xi_1}{dz}-\frac{d\zeta_1}{dx}&=\frac{\mathrm{K}_1}{\mathrm{K}}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+2\nu\frac{d\dot{\eta}}{dz}.\\ \frac{d\eta_1}{dz}&=\frac{\mathrm{K}_1}{\mathrm{K}}\cdot\frac{dx}{dz}-\nu\left(2\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(\mathrm{I})$$

while if it be all in x—*i.e.*, in the intersection of the plane of incidence and the surface, $\alpha=\mathfrak{M}$, $\beta=\gamma=0$, and $\frac{d}{dh}=\mathfrak{M}\frac{d}{dx}$, so that our equations are

$$\left.\begin{aligned}\frac{d\xi_1}{dz}-\frac{d\zeta_1}{dx}&=\frac{\mathrm{K}_1}{\mathrm{K}}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+\nu\frac{d\dot{\eta}}{dx}\\ \frac{d\eta_1}{dz}&=\frac{\mathrm{K}_1}{\mathrm{K}}\cdot\frac{d\eta}{dz}-\nu\frac{d\dot{\xi}}{dx}\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(\mathrm{II})$$

I shall now proceed to solve these two systems of equations each for the two cases of waves whose magnetic forces $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ are first in the plane of incidence, and secondly at right angles to it. From the forms of the equations it is evident that we cannot assume either the reflected or refracted rays to be similarly polarised, but it is easy to convince oneself that there can be no difference of phase introduced in this case any more than in the former one of ordinary reflection, and, as I then remarked, it is evidently a question of greater complication than to be capable of being deduced from the simple assumption that the alteration of the nature of the medium in going from one into another is abrupt. Until more is known of the nature and extent of this change I fear we must be content with theories which only partially represents the facts.

As before, I shall assume ξ_1, η_1, ζ_1 to be due to the incident and reflected rays, while ξ, η, ζ are due to the refracted ray. The incident ray may be taken to depend upon the angle

$$\phi_1=\frac{2\pi}{\lambda_1}(t-z\cos i-x\sin i)$$

calling the velocity in this medium unity and the reflected wave on

$$\phi'_1=\frac{2\pi}{\lambda_1}(t+z\cos i-x\sin i),$$

as it is easy to see that for these to vanish from the equations when $z=0$ we must have the sines of the angles of incidence and reflection equal, while the refracted wave must be taken to depend upon

$$\phi=\frac{2\pi}{\lambda_1}(st-z\cos r-x\sin r)$$

and here we must have, as before, $\frac{s}{\lambda}=\frac{1}{\lambda_1}=\frac{\sin i}{\lambda_1}=\frac{\sin r}{\lambda}$, and this involves, as before, the ordinary laws of refraction, as s is constant and in general

$$s=\sqrt{\frac{\mu_1\mathrm{K}_1}{\mu\mathrm{K}}}\ \therefore\frac{\mu_1\mathrm{K}_1}{\mu\mathrm{K}}=s^2=\frac{\sin^2 v}{\sin^2 i}$$

Taking then first the case of the magnetisation being all normal to the surface and the incident displacements ξ, η, ζ in the plane of incidence, we must evidently assume

$$\left.\begin{array}{ll}\xi_1=a_1\cos i\cos\phi_1-b_1\cos i\cos\phi'_1 & \xi=a\cos r\,.\cos\phi\\ \eta_1=c_1\sin\phi'_1 & \eta=c\sin\phi\\ \zeta_1=-a_1\sin i\cos\phi_1-b_1\sin i\,.\sin\phi'_1 & \zeta=-a_1\sin r\,.\cos\phi\end{array}\right\}\quad .\quad .\quad (\mathrm{A})$$

in order to satisfy the equations

$$\frac{d\xi_1}{dz}-\frac{d\zeta}{dx}=\frac{\mathrm{K}_1}{\mathrm{K}}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+2\nu\frac{d\dot{\eta}}{dz}$$

$$\frac{d\eta_1}{dz}=\frac{\mathrm{K}_1}{\mathrm{K}}\cdot\frac{d\eta}{dz}-\nu\left(2\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)$$

$$\xi_1=\xi\qquad\eta_1=\eta$$

As it leads to inconsistent results I, in accordance with FRESNEL, omit the third equation, $\zeta_1=\zeta$. It is not easy to justify this omission, I fear, and the result must do so, as well as the consideration that probably if we had a better insight into the nature of the change from one medium into another our equations would be so modified as not to present these anomalies.

From the last equation it is manifest that

$$c=c_1$$

and as ν is very small it is obvious that c will always be small, so that $\nu\eta$ may be omitted in the first equation. Putting in the values of ξ, η, ζ, &c., and remembering that when $z=0$, $\phi_1=\phi'_1=\phi$, these equations evidently reduce to

$$(a_1+b_1)\sin i=\frac{\mu}{\mu_1}a\sin r$$

$$c_1\,.\sin i\cos i=-\frac{\mu}{\mu_1}c_1\,.\sin r\,.\cos r-\frac{2\pi s\nu}{\lambda}a(1+\cos^2 r)\sin i$$

$$(a_1-b_1)\cos i=a\cos r$$

nd from these we must solve for a, b, and c in terms of a_1. However, c_1, the component introduced at right angles to the original plane of vibration by reflection, is the only one of much interest, though the alteration produced in the values of the reflected component by assuming $\frac{\mu}{\mu_1}$ to differ from unity are noteworthy. Calling T the period of vibration of the wave we are considering, we may put $\frac{s}{\lambda}=\frac{1}{T}$, and if we call $\frac{\mu}{\mu_1}=\chi$, we get for c the equation

$$c_1=-\frac{4\pi\nu}{T}a_1\frac{(1+\cos^2 r)\sin i\sin 2i}{(\sin 2i+\chi\sin 2r)(\sin i\cos r+\chi\cos i\sin r)}$$

which if $\chi=1$ reduces to

$$c_1=-\frac{4\pi\nu}{T}\,.\,a_1\,.\,\frac{(1+\cos^2 r)\sin^2 i\cos i}{\sin^2(i+r).\cos(i-r)}$$

In the second case, when the direction of the vibration of the incident ray is perpendicular to the plane of incidence, we must evidently assume

$$\left.\begin{array}{ll}\xi_1=-c_1\cos i\sin\phi'_1 & \xi=c\cos r\sin\phi\\ \eta_1=a_1\cos\phi_1+b_1\cos\phi'_1 & \eta=a\cos\phi\\ \zeta_1=-c_1\sin i\,.\sin\phi'_1 & \zeta=-c\sin r\,.\sin\phi\end{array}\right\}\quad .\ .\ .\ .\ .\quad (B)$$

and, as before, it is evident that c is generally very small so that $\nu\xi$ and $\nu\zeta$ may be omitted, and our equations become

$$\frac{d\xi_1}{dz}-\frac{d\xi_1}{dx}=\frac{K_1}{K}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+2\nu\frac{d\eta}{dz}$$

$$\frac{d\eta_1}{dz}=\frac{K_1}{K}\cdot\frac{d\eta}{dz}$$

$$\xi_1=\xi\qquad\eta_1=\eta$$

and when the values above are introduced into them we obtain

$$-c_1\cos i=c\cos r$$

$$(a_1-b_1)\sin i\cos i=a\chi\,.\sin r\cos r$$

$$a_1+b_1\qquad=a$$

$$-c_1=-\chi\cdot\frac{\sin r}{\sin i}c+\frac{4\pi\nu}{T}a\frac{\sin i\cos r}{\cos r}$$

Hence we get solving for c_1 in terms of a_1

$$c_1 = -\frac{8\pi\nu}{\mathrm{T}} a_1 \frac{\sin 2i \sin^2 i \cos^2 r}{\sin r(\sin 2i + \chi \sin 2r)(\sin i \cos r + \chi \sin r \cos i)}$$

which when $\chi = 1$ reduces to

$$c_1 = -\frac{4\pi\nu}{\mathrm{T}} a_1 \frac{\sin 2i \, . \sin^2 i \cos^2 r}{\sin r \sin^2 (i+r) \cos (i-r)}$$

Turning now to the case where the magnetisation is in the surface, I shall first suppose the incident vibration to be in the plane of incidence, when we evidently assume, as before, equations (A) for ξ_1, η_1, ζ_1, ξ, η, ζ, and these must satisfy the equations (II), and in addition

$$\xi_1 = \xi \qquad \eta_1 = \eta$$

As before, we evidently get $c = c_1$, and from the second equation of (II) see that c must be small, as ν is, and consequently we may omit $\nu\eta$ in the first equation, and so the first equation is the same as before, and we have

$$(a_1 + b_1) \sin i = a\chi \, . \sin r$$
$$(a_1 - b_1) \cos i = a \cos r$$
$$c_1 \sin i \cos i = -c_1 \chi \sin r \, . \cos r - \frac{2\pi\nu}{\mathrm{T}} a \cos r \, . \sin^2 i$$

which when solved for c_1 in terms of a_1 gives

$$c_1 = -\frac{4\pi\nu}{\mathrm{T}} a_1 \frac{\sin 2i \sin^2 i \cos r}{(\sin 2i + \chi \sin 2r)(\sin i \cos r + \chi \cos i \sin r)}$$

which when $\chi = 1$ reduces to

$$c_1 = -\frac{2\pi\nu \, . a_1}{\mathrm{T}} \cdot \frac{\sin 2i \, . \sin^2 i \cos r}{\sin^2 (i+r) \, . \cos (i-r)}$$

Finally, supposing the incident vibration to be perpendicular to the plane of incidence, we have ξ_1, η_1, ζ_1, ξ, η, ζ determined by the equation (B), which when substituted in equations (II), neglecting $\nu\xi$ as before on account of its smallness, give

$$c_1 = -\frac{4\pi\nu}{\mathrm{T}} a_1 \frac{\sin 2i \sin^2 i \cos r}{(\sin 2i + \chi \sin 2r)(\sin i \cos r + \chi \cos i \sin r)}$$

which when $\chi = 1$ reduces to

$$c_1 = -\frac{2\pi\nu}{\mathrm{T}} a_1 \cdot \frac{\sin 2i \, . \sin^2 i \, . \cos r}{\sin^2 (i+r) \, . \cos (i-r)}$$

It is remarkable that in this case the two components introduced by reflection are the same, whether the vibration be in, or perpendicular to, the plane of incidence.

Comparing this method of obtaining the effects of reflection from a magnetised surface with that given by me in the 'Proceedings of the Royal Society for 1876,' No. 176, it is to be observed that my equations (I) and (II) are unaltered if the signs of ν and η are both reversed together, or if those of ν and ξ and ζ are all reversed, showing that a circularly polarised ray in one direction is reflected according to the same laws when the magnetisation is one way as the oppositely circularly polarised ray would be if the magnetisation were reversed, and hence my former method of dividing the incident plane polarised ray into two opposite circularly polarised ones, each of which was reflected according to its own laws, is justified.

In comparing these expressions with the results of Mr. KERR's admirable experiments, it is necessary to observe, as I mentioned before, that the introduction of a difference of phase between the reflected components is a question of a different order from that here discussed, and probably to some extent at least depends on the want of abruptness in the change from one medium to the other. For instance, my expressions give no change of plane of polarisation when light is reflected normally from the end of a magnet, but they would lead one to expect that the only effect was a slight elliptic polarisation, the major axis of the ellipse being in the same plane as the original plane of polarisation. Now Mr. KERR's experiments show that there is some rotation of this plane by reflection, and a supposition similar to one long ago proposed to explain the known elliptic polarisation of metallic reflection—namely, that the efficient reflecting surface has some depth—may easily be shown to lead to Mr. KERR's result. On this hypothesis the reflected ray is the resultant of the rays reflected from a small thickness at the surface of separation of the media; and in the case of normal reflection from the end of the pole of a magnet, each of these components would be slightly turned from its original plane of polarisation owing to having passed through a very small thickness of a very powerful rotatory polarising substance—namely, this superficial layer of the magnet—hence it is evident that their resultant would no longer be polarised in the same plane as the incident ray. I only give this as an instance of how this question of a difference of phase affects the results, and how the hypotheses that have been framed to explain it might be used to bring my results into complete accord with Mr. KERR's experiments. I hardly think it worth while going into this more fully, as it is treading so closely upon unknown ground—namely, the connexion between matter and ether—that our hypotheses are to a great extent merely conveniences.

Another question is the extent to which χ affects ordinary reflection from a powerfully magnetic substance like iron. I have never come across any experiment tending to show that the reflection from iron was at all peculiar. This may be owing to the electrostatic inductive capacity being a characteristic of the ether in the matter, while magnetic inductive capacity is a characteristic of the matter, and so only affects the wave propagation in a secondary degree—*i.e.*, only to the extent to which ether motions are transformed into currents in or motions of the matter. The same applies to the opacity of the medium, which is similarly due to an exchange of energy between the ether and the matter. The function Professor Maxwell assumes to represent the effect of magnetisation on wave-propagation is an expression of the same hypothesis. In comparing my equations with Mr. Kerr's results I shall consequently assume $\chi=1$.

Mr. Kerr's most elaborate experiments have been made with the lines of magnetic force in the intersection of the surface of the medium and the plane of incidence,[*] and I shall confine my attention to this case. My equations give that the principal effect of reflection is to introduce a component perpendicular to the original plane of polarisation whose amplitude is represented by the equation

$$c=-\frac{2\pi\nu}{\mathrm{T}}\,\frac{\sin 2i\sin^2 i\cos r}{\sin^2(i+r)\,.\cos(i-r)}$$

when the amplitude of the incident vibration is unity.

The following table represents the values of the variable part of this for the several incidences mentioned by Mr. Kerr in his paper. This table is calculated on the assumption that 75° is the polarising angle for iron.

$i=$	0	30°	45°	60°	65°	75°	80°	85°	90°
$\frac{\sin 2i\sin^2 i\cos r}{\sin^2(i+r)\,.\cos(i-r)}=$	0	·6272	·8646	1·0007	1·0058	·9000	·7554	·4904	0

This table shows that the intensity of this introduced component vanishes for the two limiting incidences 0° and 90°, and attains a maximum value of 1·0067 at about 63° 20′. On comparing this with Mr. Kerr's results there is a most striking correspondence. Summing up the results of his paper on reflection from a surface which is magnetised so that the lines of force are in the surface and the plane of incidence (‘Phil. Mag.,’ March, 1878, § 23), he says: “When the vibration reflected from the unmagnetised mirror is either parallel or perpendicular to the plane of reflection, the effect of magnetisation is to introduce a new and very small component vibration in a direction perpendicular to the primitive vibration.” This is what I have called c. He goes on to define its direction relatively to the Amperean currents which are supposed to produce the magnetic force; but as the sign of c depends on the sign of ν, and thus on that of C, which cannot be certainly determined for iron otherwise than by these very experiments, any confirmation founded upon it such as I mentioned in my former paper is to a certain extent illusory; but the same arguments as are there used would lead to the same conclusion here—namely, that iron, if transparent, would behave like those ferromagnetic substances observed by VERDET, which rotate the plane of polarisation in the opposite direction to the Amperean currents, as is of course most probable. He further mentions, in § 24, that the phase of this component is always nearly the same as that of the component of the reflected ray polarised in the plane of incidence. This question, however, of difference of phase is one I am not at present prepared to explain, and, as I have mentioned already, must await a more comprehensive theory. Considering next the intensity, of course no very accurate measures were possible on account of the minuteness of the phenomenon; but he several times remarks that all effects of magnetisation vanished at normal and grazing instances, while his maximum effects were always obtained at incidences of about 60° or 65°. The anomalies observed connected with the incidence 75° all belong to the question of the difference of phase between the components, which my investigation does not touch. On the whole then I think that my results, as far as they go, are in complete accordance with Mr. KERR's experiments.

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[*] ‘Phil. Mag.,’ March, 1878.

This investigation is put forward as a confirmation of Professor MAXWELL's electromagnetic theory of light, in which, though there are some points requiring further investigation, nevertheless the foundation has certainly been laid of a very great addition to our knowledge, and if it induced us to emancipate our minds from the thraldom of a material ether might possibly lead to most important results in the theoretic explanation of nature.